The Cow/Goat Problem

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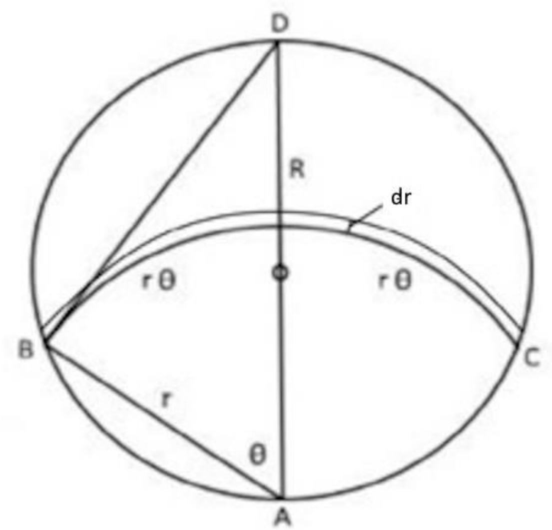
A cow (or goat) is attached by a rope to a point on the perimeter of a circular field. How long should the rope be in terms of the radius of the field so that the cow can graze in exactly half of the field?

This problem has been around in one form or another for a long time. You can find many solutions to it on the internet. But all of the solutions I've seen are geometric solutions, using plane geometry. They all chop up the pasture into various shapes and add up the area of each one.

I have found a solution using calculus that I think is a lot simpler and more elegant. This solution defines a differential of area that is centered around the point on the perimeter of the pasture where one end of the rope is tied. Then it integrates from the origin (zero) until the rope is long enough to allow the cow to graze on half the area of the circular pasture. Here's the derivation:

Let R be the radius of the cow pasture. Let r be the length of the rope.
Let Θ be the angle between a line from the point where the rope is tied on the perimeter of the pasture to the center of the pasture, and a line of length r from where the rope is tied on the perimeter to the edge of the pasture. See diagram:



1. BC is a circular arc of radius r centered at point A.
   The length, L, of BC is $2\Theta r$.

2. $\cos(\Theta) = AB/AD = r/(2R)$, so $\Theta = \cos^{-1}(r/(2R))$. (Note: Angle ABD is a right angle because it intercepts a 180 degree arc.)

3. So $L = 2\Theta r = 2r * \cos^{-1}(r/(2R))$.

4. Consequently, one form of a differential of area of the pasture is

$$dArea = (2r * \cos^{-1}(r/(2R)))dr.$$

We want to integrate this differential along AD starting at point A until r is long enough so that half the area of the pasture is enclosed, so we need to solve for x:

$$Area = \int (2r \cdot \cos^{-1}(r/(2R)))dr = \pi(R^2)/2.$$

WolframAlpha chokes on this integral, but easily solves it if we let R = 1

$$Area = \int (2r \cdot \cos^{-1}(r/2))dr = \pi/2.$$

Letting R = 1 will tell us what the ratio of r to R should be. Bringing up WolframAlpha and using this as the input:

solve(integrate((2r)arccos(r/2))dr = π/2)

WolframAlpha gets:

((4 - r^2)^(1/2))r + pi = 2(r^2)arccos(r/2) + 4(arcsin)(r/2)

Solving this equation for r, WolframAlpha gets:

r = 1.15872847301812...

So the rope should be so that the cow can graze in exactly half of the field is
R * 1.15872847301812 to 14 decimal places.