Mathematics Higher Level Internal

Assessment Exploration

“Sound Waves Reflected by the

Auricle”

Examination Session: May 2020

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Rationale

From a young age, I have always been curious about the peculiar shape of the auricle (pinna), or the outer ear, and wondered what its purpose and function was. Knowing the human body, I knew it must have a purpose, or we would have lost it after millions of years of evolution. Upon doing some research, I learned that the auricle was specifically designed to focus and amplify sound waves into the ear canal. It is not required, but its removal would somewhat diminish our hearing ("The Outer Ear - Functions & Parts of the Outer Ear | Pinna | Eardrum - Hear-It.Org").

Introduction

Sound travels as longitudinal pressure waves created by vibrating objects. In the microscopic scale, a vibrating object first pushes the medium (gas, liquid, or solid) forward, creating regions of high density, the compressions then pulls the medium back, creating regions of low density, the rarefactions. When these compressions and rarefactions reach our ears, we hear them as sound. Because the molecules move in the same direction as the sound wave, this is a longitudinal wave ("Sound Waves and Music").

Sound is a wave, so it can be reflected. This occurs when a wave reaches a boundary between two different mediums and returns to the original medium. For the purpose of this investigation, I will only consider reflected sound waves and will use the Law of Reflection, stating the angle of incidence is equal to the angle of reflection.

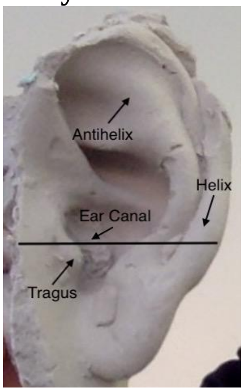
Aim

The aim of this investigation is to determine how effective my auricle is at amplifying sound, measured by what percent of sound waves are reflected into the ear canal. I will use functions to graph ear cross sections, matrices to calculate these functions, linear functions, 2D and 3D vectors and planes to calculate sound wave reflections, differentiation to calculate the slope of the functions, and integration to calculate the area and volume of the ear midsection.

Methodology

Everybody's ear shape and size are different, so in this investigation, I will use my own ear to make clay models. First, I sliced it as shown in Figure 1, graphed the shape of the cross section, and labelled some of the ear parts (R.Shah) as shown in Figure 2. I calculated how sound waves reflected into the ear canal using this 2D model, linear functions, and vectors. Then, I graphed 5 horizontal and 3 vertical cross sections to create an approximate 3D ear model and used 3D vectors to calculate the reflections.

Clay Ear Model



Stamped Horizontal Cross Section

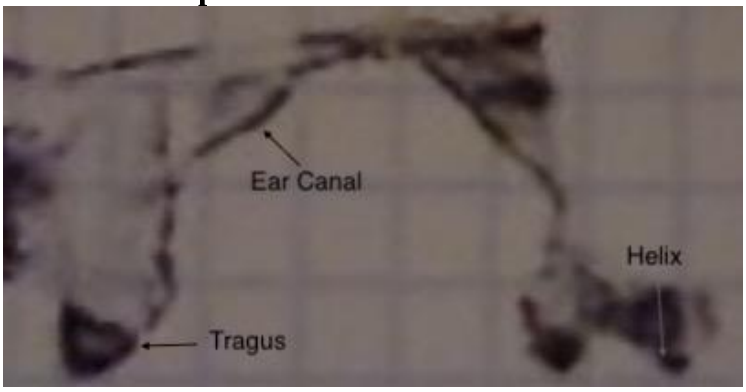
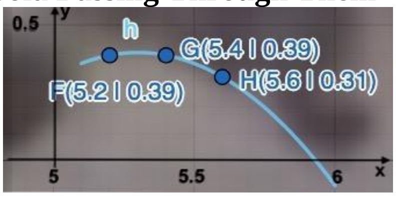


Figure 1. Ear Model 1 (Self) **Figure 2. Ear cross section 4.2 cm from the top (Self)**

Graphing My Horizontal Cross Section

3 Chosen Points and Parabola Passing Through Them

Graph A. 3 Points



Graph B. Quadratic Function

First, I graphed the rightmost section using the three points:

$F = (5.2, 0.39)$, $G = (5.4, 0.39)$ and $H = (5.6, 0.31)$

as shown in Graph A. The ordinate of the two points, F and G are equal, and with point H, the curve is concave down so I can approximate this function to be quadratic. I wrote a system of three equations using the standard form $y = ax^2 + bx + c$ where a, b, and c are real numbers and x is the variable. I substituted the x and y values of the three points as shown below. The three equations can be separated into three matrices by the equal sign and by parameters and variables. This is possible because matrix multiplication is solved using the dot product of each row of the first matrix and each column of the second matrix ("Solving Systems of Linear Equations Using Matrices").

$$27.04a + 5.2b + c = 0.39$$

$$29.16a + 5.4b + c = 0.39 \qquad \begin{bmatrix} 27.04 & 5.2 & 1 \\ 29.16 & 5.4 & 1 \\ 31.36 & 5.6 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.39 \\ 0.39 \\ 0.31 \end{bmatrix}$$

$$31.36a + 5.6b + c = 0.31$$

Matrix division is not possible, so I need to find the inverse of the 3x3 matrix and multiply it by the 3x1 matrix on the right ("Solving a 3 x 3 System of Equations Using the Inverse"). When the values were simplified to 3 significant figures, the resulting determinant was 0. This would mean the matrix is not invertible and there are no solutions to the system of equations, which is not true in my case. Therefore, I did not simplify the values in my calculations. $\begin{bmatrix} 27.04 & 5.2 & 1 \\ 29.16 & 5.4 & 1 \\ 31.36 & 5.6 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.39 \\ 0.39 \\ 0.31 \end{bmatrix}$

$$27.04 \times \begin{bmatrix} 5.4 & 1 \\ 5.6 & 1 \end{bmatrix} - 5.2 \times \begin{bmatrix} 29.16 & 1 \\ 31.36 & 1 \end{bmatrix} + 1 \times \begin{bmatrix} 29.16 & 5.4 \\ 31.36 & 5.6 \end{bmatrix}$$

$$= 27.04 \times (-0.2) - 5.2 \times (-2.2) + 1 \times (-6.048) = -\frac{2}{125}$$

The method to calculate the determinant is shown above. Each value in the first row is multiplied by the 2x2 matrix composed of the remaining four digits with the second product negative. The 2x2 matrix is solved by subtracting the product of the 2nd and 3rd values from the product of the 1st and 4th values.

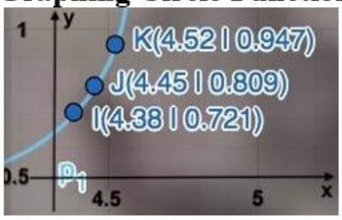
$$\begin{bmatrix} +\begin{bmatrix} 5.4 & 1 \\ 5.6 & 1 \end{bmatrix} & -\begin{bmatrix} 29.16 & 1 \\ 31.36 & 1 \end{bmatrix} & +\begin{bmatrix} 29.16 & 5.4 \\ 31.36 & 5.6 \end{bmatrix} \\ -\begin{bmatrix} 5.2 & 1 \\ 5.6 & 1 \end{bmatrix} & +\begin{bmatrix} 27.04 & 1 \\ 31.36 & 1 \end{bmatrix} & -\begin{bmatrix} 27.04 & 5.2 \\ 31.36 & 5.6 \end{bmatrix} \\ +\begin{bmatrix} 5.2 & 1 \\ 5.4 & 1 \end{bmatrix} & -\begin{bmatrix} 27.04 & 1 \\ 29.16 & 1 \end{bmatrix} & +\begin{bmatrix} 27.04 & 5.2 \\ 29.16 & 5.4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -0.2 & 2.2 & -6.05 \\ 0.4 & -4.32 & 11.6 \\ -0.2 & 2.12 & -5.62 \end{bmatrix}; \begin{bmatrix} -0.2 & 0.4 & -0.2 \\ 2.2 & -4.32 & 2.12 \\ -6.05 & 11.6 & -5.62 \end{bmatrix}$$

To calculate the inverse, the new value in row a column b is the determinant of the 4x4 matrix found by crossing out row a and column b in the original matrix as shown above. The inverse is found by multiplying this matrix with the inverse of the determinant. The result is

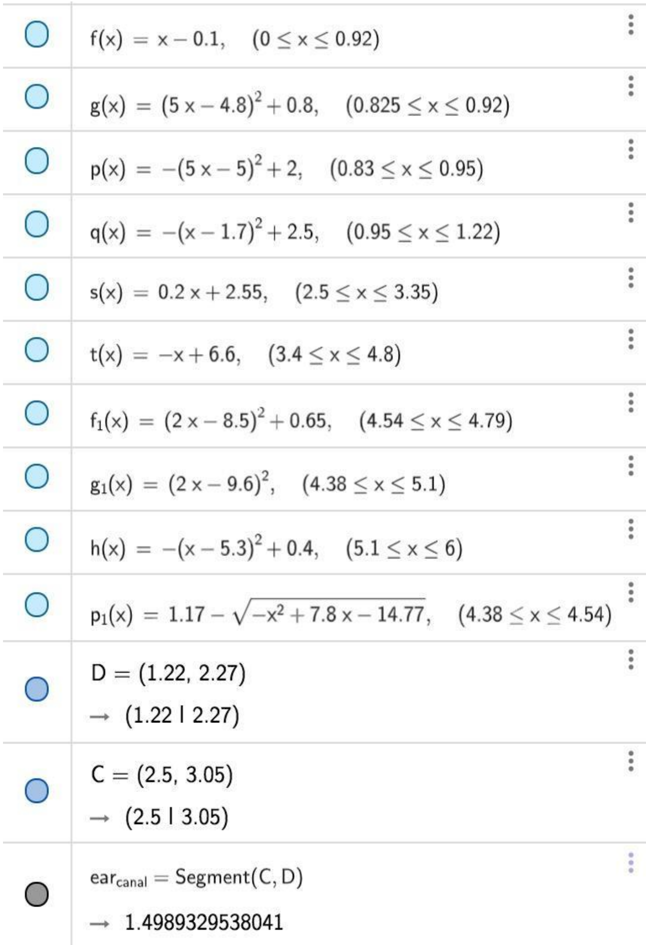
$$-\frac{125}{2}\times\begin{bmatrix}-0.2 & 0.4 & -0.2\\ 2.2 & -4.32 & 2.12\\ -6.05 & 11.6 & -5.62\end{bmatrix}\times\begin{bmatrix}0.39\\0.39\\0.31\end{bmatrix}=\begin{bmatrix}-1\\10.6\\27.7\end{bmatrix}$$

Once the calculations were complete, I rounded the values of a, b, and c to 3 significant figures and the equation of the parabola is $g(x) = -x^2 + 10.6x + 27.7$. Taking the square, I got the quadratic equation $g(x) = -(x-5.3)^2 + 0.4$ whose domain I restricted to $5.1 \le x \le 6$. Another example is a circle function passing through points $I = (4.38, 0.721), J = (4.45, 0.809)$ and $K = (4.52, 0.947)$ as shown in Graph C. I can substitute these values into the circle equation $x^2 + y^2 + 2gx + 2fy + c = 0$ (where the x and y are the variables and g, f, and c are the parameters) and solve the system of equations. Rearranging the values and approximating them to 3 significant figures, the function is $(x-3.9)^2 + (y-1.17)^2 = 0.437$. I expressed this equation in terms of y as the half circle function $p_1(x) = 1.17 - \sqrt{-x^2 + 7.8x - 14.77}$ to restrict its domain.

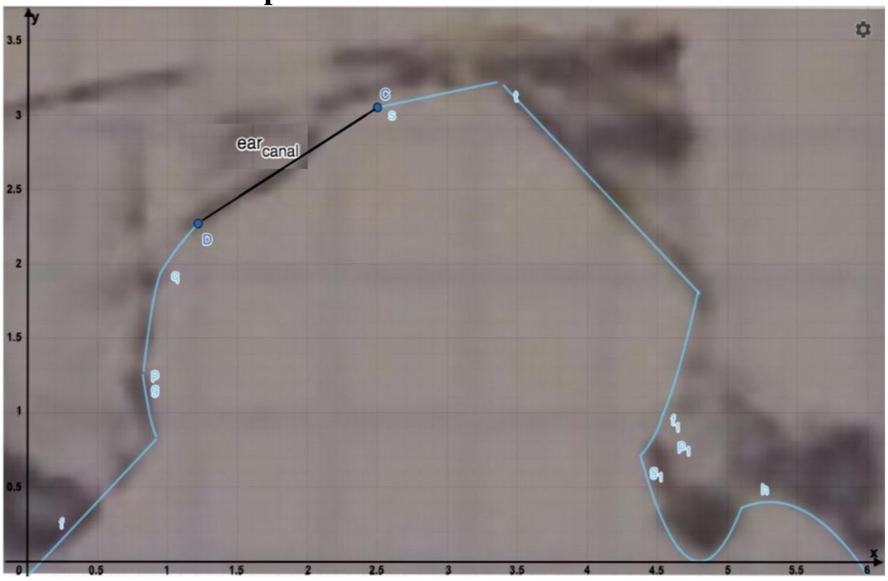
Graphing Circle Function



Graph C. Circle Function



Graphed Horizontal Cross Section



Graph D. Piecewise Functions to Graph Ear Cross Section.

All other functions were calculated using a similar technique and graphed as shown in Graph D. The black line segment CD on the graph represents the ear canal, 1 cm in length and located 2.3 cm from the right end of the ear. The calculations to find the derivatives of each function using the power rule, chain rule, and implicit differentiation are shown below.

$$f'(x) = 1,\ p'(x) = -50x + 50,\ q'(x) = -2x + 3.4,\ s'(x) = 0.2,\ t'(x) = -1,$$

$$f_1{}'(x) = 8x - 34,\ g_1{}'(x) = 8x - 38.4,\ h'(x) = -2x + 10.6$$

$g'(x) = ((5x - 4.8)^2 + 0.8))' = 2 \times (5x - 4.8) \times (5x - 4.8)' + (0.8)' = 50x - 48$ using the chain rule and power rule.

$$p_1'(x) = \left(1.17 - \sqrt{-x^2 + 7.8x - 14.77}\right)'$$

$$= 0 - \frac{1}{2}(-x^2 + 7.8x - 14.77)^{-\frac{1}{2}} \times (-x^2 + 7.8x - 14.77)'$$

$$= \frac{x - 3.9}{\sqrt{-x^2 + 7.8x - 14.77}}$$

I can also find the derivative of function $p_1(x)$using implicit differentiation.

$$\frac{d}{dx}[(x - 3.9)^2 + (y - 1.17)^2] = \frac{d}{dx}[0.437]$$

$$2 \times (x - 3.9) \times \frac{d}{dx}(x - 3.9) + 2 \times (y - 1.17) \times \frac{d}{dx}(y - 1.17) = 0$$

$$(2x - 7.8) + (2y - 2.34)\frac{dy}{dx} = 0;\ \frac{dy}{dx} = \frac{-2x+7.8}{2y-2.34}$$

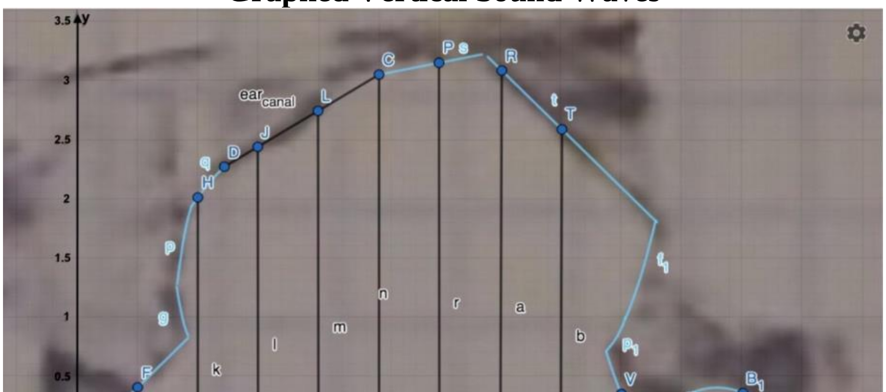
At point $\left(4{,}1.17 - \sqrt{-(4)^2 + 7.8 \times (4) - 14.77}\right) = (4{,}0.514)$, the two derivatives are equal

$p'(4) = \frac{4-3.9}{\sqrt{-4^2+7.8\times4-14.77}} = 0.152$, $\frac{dy}{dx} = \frac{-2\times4+7.8}{2\times0.514-2.34} = 0.152.$

When sound is emitted from a source, it moves outward in all directions in the form of a spherical wave. For simplicity, I will assume the sound source is located far away so the sound waves can be drawn as planar waves, straight lines that are parallel to the x axis. I will calculate the echoes at 12 different locations and represent different sections of the sound wave as vertical rays as shown in Graph G. While each vertical ray represents a section on a sound wave front, I will refer to them as sound waves for simplicity.

Method 1

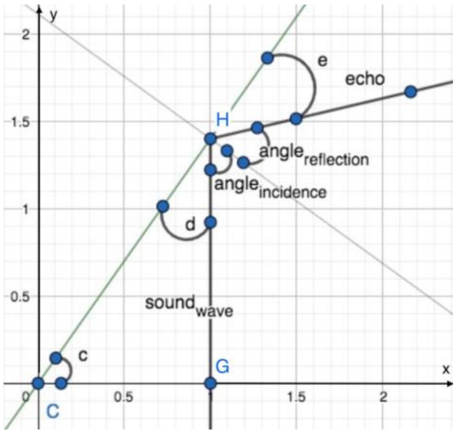
Graphed Vertical Sound Waves



Graph E. One wave front of an incoming sound wave

I will calculate the reflection of sound wave HG, shown in Graph E above. Sound wave HG intersects function $q(x) = -(x - 1.7)^2 + 2.5$ at the point x $= 1$, where the y value is equal to $q(1) = -(1 - 1.7)^2 + 2.5 = 2.01$ and slope is $q'(1) = -2 \times 1 + 3.4 = 1.4$.

Sound Wave and its Reflection



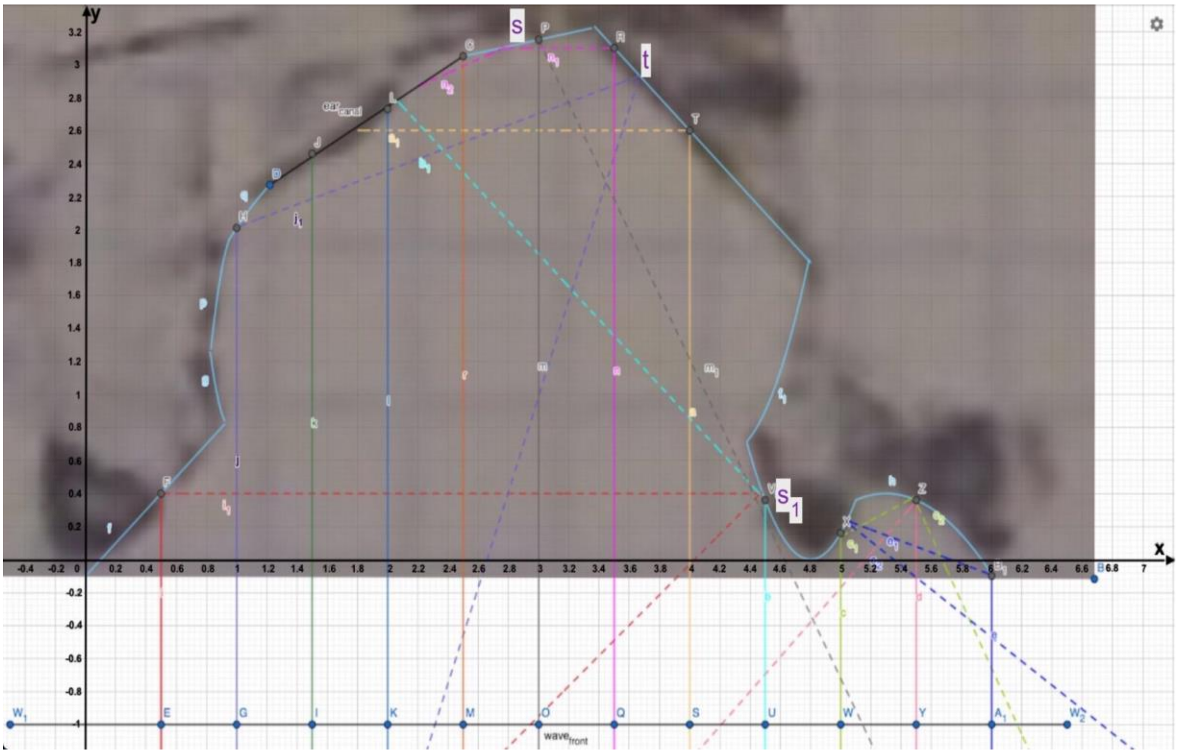
Graph F. Angle of incidence and reflection of sound wave HG

Graph F above shows the slope of the ear function at point of intersection (green line), the normal, perpendicular to the ear slope (grey line), the sound wave, and the echo. Triangle ABC is a right triangle because the sound wave was chosen to be perpendicular to the x axis. $\angle d = 90^o - \angle c$, where $\angle c$ is the angle between the ear slope and the x axis. From the slope, $\measuredangle c = \arctan(1.4) = 54.5^o$so $\measuredangle d = 90^o - 54.5^o = 35.5^o$.Therefore, the angle of incidence is equal to $90^o - 35.5^o = 45.5^o$, which is always equal to the angle of reflection by the Law of Reflection("The Law of Reflection"). To find the slope of the echo, I need to find the angle it makes with the x axis. Let's call this angle $\angle f$. If a line parallel to the x axis was drawn passing through point A, the angle between the ear slope and the parallel line will be equal to $\angle c$, from corresponding angles. Then, $\angle f = \angle c - \angle e$. Because the normal is perpendicular to the ear slope and the angle of incidence is equal to the angle of reflection, $\angle d = \angle e$. This means angle $\measuredangle f = 54.5^o - 35.5^o = 19^o$. Therefore, the slope of the line representing the echo is $\tan(19^o) = 0.344$.

Substituting into linear equation $(y - y_o) = m(x - x_o)$, $(y - 2.01) = 0.344 \times (x - 1)$ the linear function of the reflected sound wave is $f_1(x) = 0.344x - 1.67$ which then intersects function $t(x) = -x + 6.6$. Using the same technique once more, I found the slope of the second echo is $\tan(71^o) = 2.90$ and that it passes through point $(3.67, 2.93)$. From linear equation $y = mx + b$ where m is slope and b is y intercept, $b = y - mx = 2.93 - 2.90 \times 3.67 = -7.71$

and the equation of the line is $j_2 = 2.90x - 7.71$. Function j_2 does not intersect the line segment CD, so the sound wave will not enter the ear canal. The sound wave and its reflections are the purple linear functions represented by the j, j_1, j_2in Graph G below. All other sound wave reflections were calculated using the same method and are graphed below using different colors.

Vertical Sound Waves and Their Reflections



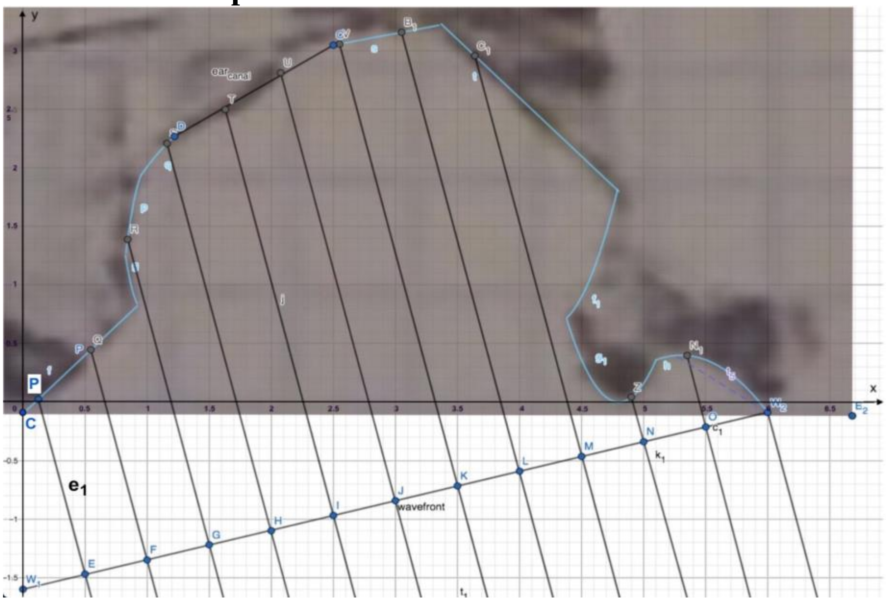
Graph G. Echoes of all sound waves represented by the dashed lines.

This method was effective and 6 out of 12 sound waves entered the ear canal. Every second, thousands of sound waves reach our ears, and our ears are imperfect, so this result is plausible. As can be seen on Graph G, sound waves either entered directly or reflected off function t, s, or s_1.From this result, I predict there are specific sections of the auricle that reflect sound waves into the ear canal depending on the location of the sound waves. This result supports my prediction that the auricle funnels incoming sound waves into the ear canal.

However, this is only one horizontal cross section, does not consider the vertical slope of the auricle and only uses 12 vertical sound waves. To expand this project within the page limit, I will try a second method with sounds waves coming at a different angle. Furthermore, the technique I used to calculate the reflections was time consuming, with many calculations, and prone to errors. To simplify my calculations, I will use vectors and to make my investigation constant, I will use twelve parallel sound waves.

Method 2

Graphed Non-Horizontal Sound Waves

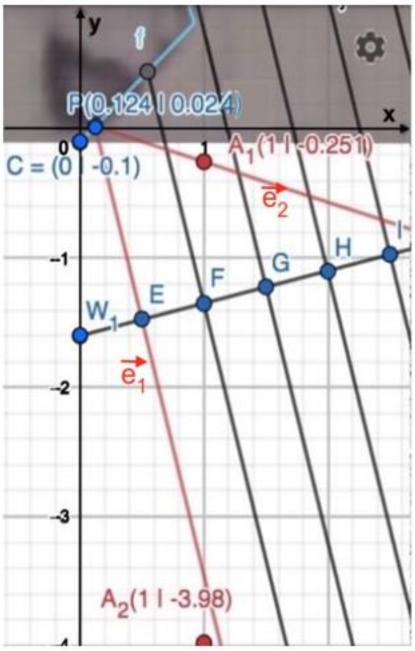


Graph H. Wave front of sound wave at an angle

I can represent each sound wave using vectors, as shown in Graph H. First, sound wave $\vec{e_1}$ has two points, $E = (0.5, -1.47)$ and $P = (0.124, 0.024)$. The vector equation is written using a point vector $\overrightarrow{OE} = \begin{pmatrix} 0.5 \\ -1.47 \end{pmatrix}$ and direction vector $\overrightarrow{EP} = \overrightarrow{OP} - \overrightarrow{OE} = \begin{pmatrix} 0.124 - 0.5 \\ 0.024 + 1.47 \end{pmatrix} = \begin{pmatrix} -0.376 \\ 1.49 \end{pmatrix}$. $\vec{e_1} = \begin{pmatrix} 0.5 \\ -1.47 \end{pmatrix} + \begin{pmatrix} -0.376 \\ 1.49 \end{pmatrix} \lambda$. Likewise, I can represent the linear function f(x) as a vector with position vector $\overrightarrow{OC} = \begin{pmatrix} 0 \\ -0.1 \end{pmatrix}$ and direction vector $\overrightarrow{CP} = \begin{pmatrix} 0.124 \\ 0.124 \end{pmatrix}$. $\vec{f} = \begin{pmatrix} 0 \\ -0.1 \end{pmatrix} + \begin{pmatrix} 0.124 \\ 0.124 \end{pmatrix} \mu$. To calculate the equation of the echo, I need to find the angle between $\vec{e_1}$ and $\vec{f}$ using their two direction vectors, $\begin{pmatrix} -0.376 \\ 1.49 \end{pmatrix} \lambda$ and $\begin{pmatrix} 0.124 \\ 0.124 \end{pmatrix} \mu$. Using $\cos(\theta) = \frac{\vec{e_1} \cdot \vec{f}}{|\vec{e_1}| \times |\vec{f}|} = \frac{(-0.376 \times 0.124) + (1.49 \times 0.124)}{\sqrt{0.376^2 + 1.49^2} \times \sqrt{0.124^2 + 0.124^2}} = 0.513$, $\theta = arccos(0.513) = 59.1^o$. The angle of incidence is $90^o - 59.1^o = 30.9^o$, which is equal to the angle of reflection. Therefore, I need to find vector $\vec{e_2}$ that makes 59.1^o angle with vector

$\vec{f} = \begin{pmatrix} 0 \\ -0.1 \end{pmatrix} + \begin{pmatrix} 0.124 \\ 0.124 \end{pmatrix}\mu$, 61.8^o degree angle with vector $\overrightarrow{e_1} = \begin{pmatrix} 0.5 \\ -1.47 \end{pmatrix} + \begin{pmatrix} -0.376 \\ 1.49 \end{pmatrix}\lambda$, passes through point $P = (0.124, 0.024)$ and has a direction vector $\begin{pmatrix} 1 \\ b \end{pmatrix}$. I chose the first term of the direction vector as 1 because with two unknowns, there is an infinite number of vector solutions due to parallel vectors. Solving the equation $\cos(59.1^o) = \frac{0.124+0.124b}{\sqrt{1+b^2}\times\sqrt{0.124^2+0.124^2}}$, we get the quadratic equation $0.00728b^2 + 0.0308b + 0.00728 = 0$. There are two solutions, $b = -3.98$ and $b = -0.251$. Graphing them, as shown in Graph I, $b = -3.98$ is implausible because the angle of incidence is always equal to the angle of reflection meaning the slope of the reflection should be less steep than the slope of the sound wave. The solution is $b = -0.251$. With position vector $\overrightarrow{OP} = \begin{pmatrix} 0.124 \\ 0.024 \end{pmatrix}$ and direction vector $\begin{pmatrix} 1 \\ -0.251 \end{pmatrix}$, $\overrightarrow{e_2} = \begin{pmatrix} 0.124 \\ 0.024 \end{pmatrix} + \begin{pmatrix} 1 \\ -0.251 \end{pmatrix}\delta$. I can check that this vector makes a 61.8^o angle with the vector $\overrightarrow{e_1} = \begin{pmatrix} 0.5 \\ -1.47 \end{pmatrix} + \begin{pmatrix} -0.376 \\ 1.49 \end{pmatrix}\lambda$. Using the dot product, $\cos(\theta) = -0.473$. $\theta = 180 - \arccos(-0.473) = 61.8^o$. This answer is an exact match, meaning vector $\overrightarrow{e_2}$ accurately represents the reflection of the sound wave.

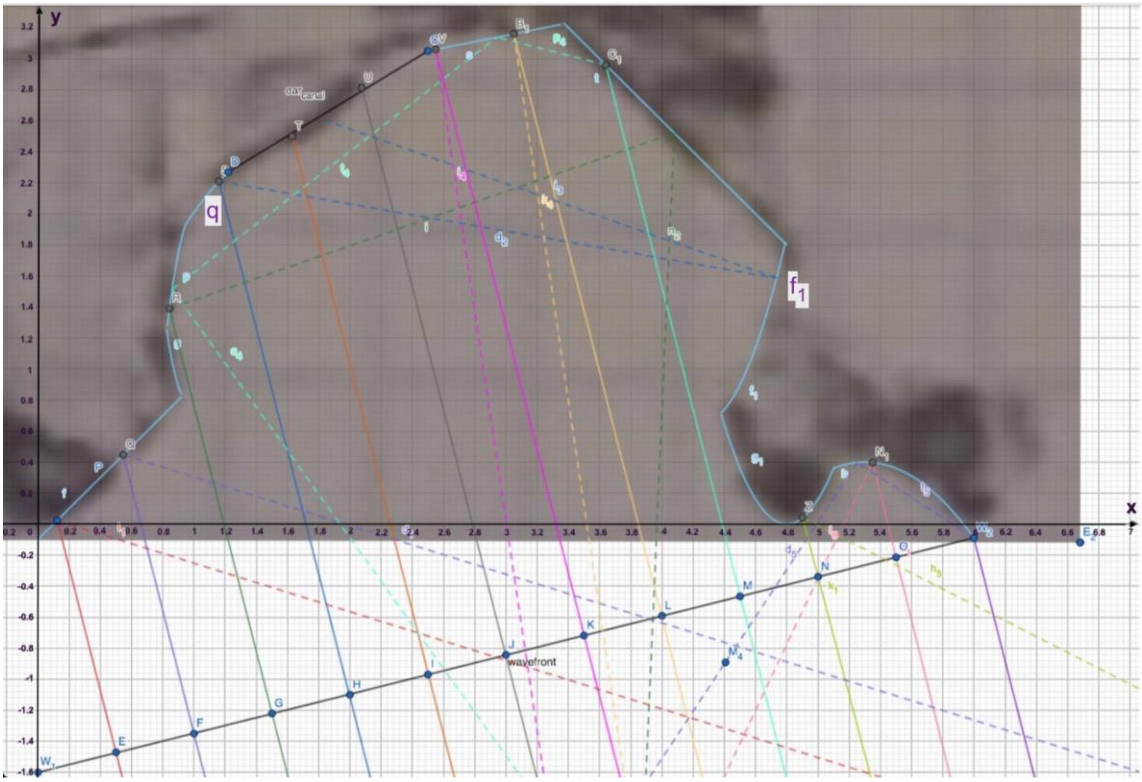
Two Solutions A1 and A2



Graph I. Two Solutions

Likewise, all reflections have been calculated as shown in Graph J. Here, 3 out of 12 sound waves entered the ear canal, which is a smaller percent than before. But looking more closely, sound waves that entered the ear canal either went straight in or reflected off functions q and f_1, two different regions of the ear than in my first method. From this, I predict the auricle has different regions to funnel sound waves into the ear canal depending on the sound waves' location and direction. I also predict different horizontal cross sections have different shapes to reflect different sound waves into the ear canal, making the auricle effective.

Non-Horizontal Sound Waves and Their Reflections

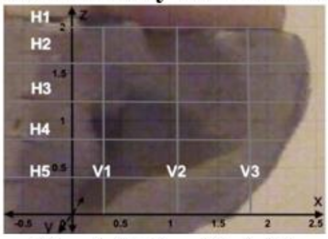


Graph J. Reflections of all 12 sections of the sound wave front.

It is possible to continue this analysis using the same ear model and calculating the reflections of sound waves coming from different angles. I can also graph different horizontal or vertical cross sections of the ear. However, these are not effective models of the ear because the ear is a complex 3D model that cannot be completely analyzed using 2D cross sections. For instance, sound waves that enter the ear canal may reflect too high or low to enter the ear canal and sound waves that seem to miss the ear canal may reflect off different sections of the ear and enter the ear canal.

Method 3

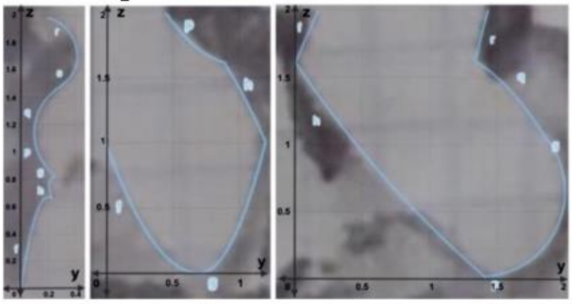
Second Clay Ear Model



Graph K. Ear Model 2

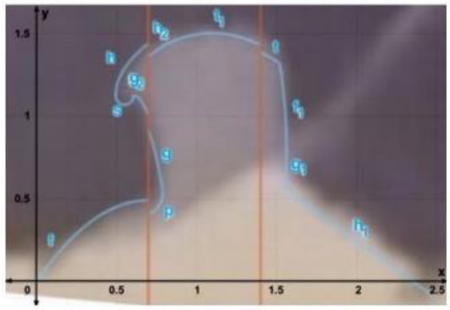
For my third method, I will create a 3D model of a 2 cm tall midsection of my ear using 5 horizontal and 3 vertical cross sections as shown in Graph K. To make an accurate model I

Graphed Vertical Cross Sections



Graph L. V1 Graph M. V2 Graph N. V3

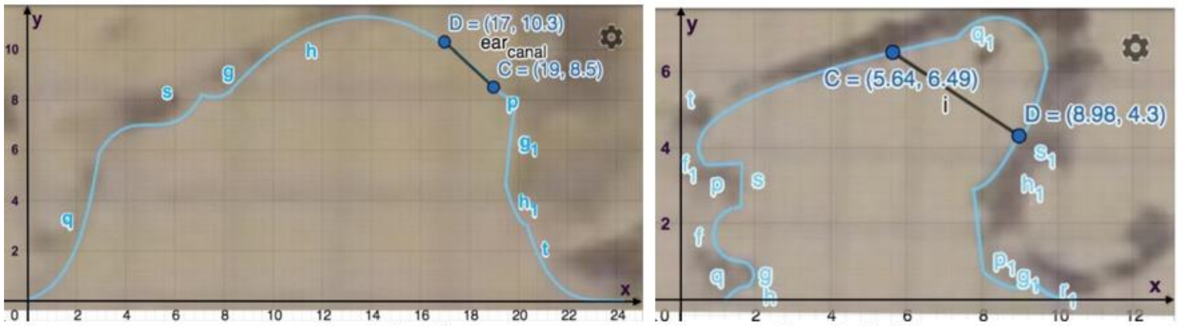
Shifted Horizontal Cross Section



Graph O. Horizontal Cross Section z=0.3

calculated the change in y value of my ear from the 3 vertical functions and used this value to translate my horizontal cross sections in the y direction. I used the value from function V1, Graph L for the range $0 \leq x \leq 0.7$, function V2, Graph M for range $0.7 \leq x \leq 1.4$ and function V3, Graph N for range $1.4 \leq x \leq 2.5$. The translated horizontal cross sections are 1mm apart and I combined them into one 3D piecewise function. For example, in range $0 \leq z \leq 0.4$, I used the function H5. For range $0.2 \leq z \leq 0.3$ I need to calculate how the y value of the three vertical functions changed to vertically shift the three sections of function H5. The function in V1 corresponding to $z = 0.4$ is $f(y_1) = 0.4$; Therefore, from $-17(y_1 - 0.2)^2 + 0.67 = 0.4$, $y_1 = 0.0740$. $y_2 = 0.0525$ was calculated the same way and their difference is $y_2 - y_1 = -0.0215$. Thus, the function is vertically transformed by -0.02151 in range $0 \leq x \leq 0.7$ shown in Graph O. The changes were also calculated for V2 and V3. The ordinate values of the functions at either side of $x = 0.7$ and $x = 1.4$ do not coincide. Therefore, the 3D model is not accurate and reflections of the sound wave at these points cannot be calculated. Furthermore, the five horizontal cross sections do not coincide, and the transition will not be smooth. However, for my investigation, this model will suffice.

Horizontal Cross Sections With Ear Canal



Graph P. H3 **Graph Q. H4**

Next, I calculated two plane equations to represent the ear canal using functions H3, Graph P and H4, Graph Q above. The two direction vectors for plane c_3 is $\overrightarrow{DD_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as the function if vertically shifted, $\overrightarrow{DC} = \begin{pmatrix} 2 \\ -1.8 \\ 0 \end{pmatrix}$ from points D and C on Graph R, and the coordinates of point C. The function H3 is 9 times larger and H4 is 4 times larger than my actual ear size, so the point coordinates were divided by 9 and 4 respectively. The plane equation is $c_3 = \begin{pmatrix} 1.9 \\ 1.14 \\ 1.2 \end{pmatrix} + \begin{pmatrix} 1 \\ -0.9 \\ 0 \end{pmatrix}\gamma + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\lambda_1$ and solving the system of equations $\begin{matrix} x = 1.9 + \gamma \\ y = 1.14 - 0.9\gamma \\ z = 1.2 + \lambda_1 \end{matrix}$ for γ and λ_1, it can be written in Cartesian form $-0.9x - y = -2.85$. The ear canal is on the left half of the ear, so the function H3 and H4 are reflected about $\text{x} = 12.1$ and $x = 5.1$, their respective midpoints. For c_3, the x value is in range $\frac{24.2-19}{9}$ and $\frac{24.2-17}{9}$, $0.578 \leq x \leq 0.8$, the y value is in range $\frac{8.5}{9}$ and $\frac{10.3}{9}$, $0.944 \leq y \leq 1.14$, and the z value is in range $0.8 \leq z \leq 1.2$. The equation of the second plane is $-0.66x - y = -2.54$ in range $0.255 \leq x \leq 1.09$, $1.08 \leq y \leq 1.62$, and $0.4 \leq z \leq 0.8$.

Integration to Calculate Area

$f(x) = \frac{x^2}{2}, \quad (0 \leq x \leq 3)$

$g(x) = -(x-4)^2 + 5.5, \quad (3 \leq x \leq 4)$

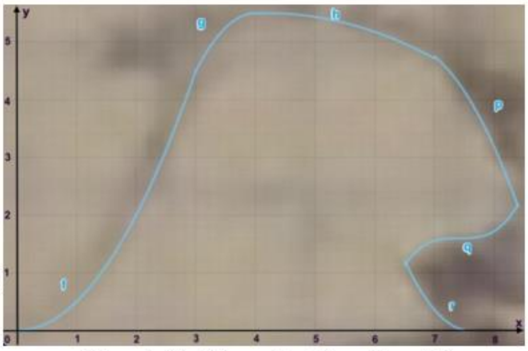
$h(x) = -\frac{(x-4.2)^2}{10} + 5.5, \quad (4 \leq x \leq 7)$

$p(x) = -(0.85x - 5.45)^2 + 5, \quad (7 \leq x \leq 8.4)$

$q(x) = (0.85x - 6.3)^3 + 1.6, \quad (6.5 \leq x \leq 8.4)$

$r(x) = (x - 7.6)^2, \quad (6.5 \leq x \leq 7.5)$

Horizontal Cross Section H1



Graph R. Piecewise Function H1

The area of piecewise function H1, shown in Graph R, can be found by adding the integral of functions $f(x)$, $g(x)$, h(x), p(x), and r(x) and subtracting the integral of function q(x). All functions are polynomial, which can be solved using polynomial integration and a substitution. To make my work concise, I will show my work for function $p(x) = -(0.85x - 5.45)^2 + 5$. First, I will use linearity to separate the integral

$$\int_7^{8.4} (-(0.85x - 5.45)^2 + 5)dx = \int_7^{8.4} -(0.85x - 5.45)^2 dx + \int_7^{8.4} 5\, dx$$

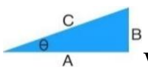
and use substitution $u = 0.85x - 5.45$; $\frac{du}{dx} = \frac{d}{dx}(0.85x - 5.45)$; $dx = \frac{20}{17}du$. Then, I calculated the new upper and lower bound values $u_l = 0.85 \times 7 - 5.45 = 0.5$, $u_u = 1.69$. $-\frac{20}{51}[u^3]_{0.5}^{1.69} + [5x]_7^{8.4} = -\frac{20}{51}(1.69^3 - 0.5^3) + (5 \times 8.4 - 5 \times 7) = 5.16$. The integral of u^2 is $\frac{u^3}{3}$ and the integral of 5 is $5x$. The result is 5.16 rounded to 3 significant figures.

To make my investigation concise, I will show my work for integrating each new type of function I used. Next is my work for integrating $q(x) = \frac{e^x}{3} - 0.3$ in function H3. The function $\int_0^{2.9} \left(\frac{e^x}{3} - 0.3\right) dx$ is linearly separated into two integrals, the multiple $\frac{1}{3}$ taken out, and the boundary values are substituted in for x. The result is 4.85 rounded to 3 significant figures

$$\frac{1}{3}\int_0^{2.9} e^x dx - \int_0^{2.9} 0.3 dx = \frac{1}{3}[e^x]_0^{2.9} - [0.3x]_0^{2.9} = \frac{e^{2.9}}{3} - \frac{e^0}{3} - 0.3 \times 2.9 + 0 = 4.85$$

Next is my work for integrating $h(x) = -\sqrt{0.13 - (x - 1.6)^2} + 0.65$ of function H4.

First, I applied linearity and represented the functions using integers. $\int_{1.8}^{1.96} -\sqrt{0.13-(x-1.6)^2}+0.65 = -\frac{1}{10}\int_{1.8}^{1.96}\sqrt{13-4(5x-8)^2}dx + \frac{13}{20}\int_{1.8}^{1.96} 1\,dx$. I used substitution $u = 5x-8$; $dx = \frac{1}{5}du$ and changed the boundary values $u_l = 1$ and $u_u = 1.8$. Then, I performed the trigonometric substitution $u = \frac{\sqrt{13}\sin(v)}{2}$; $du = \frac{\sqrt{13}\cos(v)}{2}dv$; $v = \arcsin\left(\frac{2u}{\sqrt{13}}\right)$; $\int_{1.8}^{1.96}\sqrt{13-4(5x-8)^2}dx = \frac{1}{5}\int_{1}^{1.8}\sqrt{13-4u^2}du$; $v_l = \arcsin\left(\frac{2}{\sqrt{13}}\right)$ and $v_u = \arcsin\left(\frac{3.6}{\sqrt{13}}\right)$ so $\frac{1}{10}\int_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)}\sqrt{13}\cos(v)\sqrt{13-13\sin^2(v)}dv$. Using the Pythagorean identity, I can simplify the resulting equation to $\frac{13}{10}\int_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)} cos^2(v)dv$. While the expression is much simpler, there is no direct way to solve the integral of a trigonometric function raised to a power. Usually, I would need to use the reduction formula. But here, the power is only 2, so I may use a simpler method. Rearranging the cos double angle formula $\cos(2x) = 2\cos^2(x)-1$, I can write $\cos^2(x) = \frac{\cos(2x)+1}{2}$. $\int\cos(2x)\,dx = \frac{\sin(2x)}{2}+c$, so I can use the sin double angle formula $\sin(2x) = 2\sin(x)\cos(x)$ and write this expression as $\frac{13}{10}\int_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)} cos^2(v)dv = \left[\frac{13\cos(v)\sin(v)}{10}\right]_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)} + \frac{13}{10}\int_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)}\frac{1}{2}dv$ which is equal to $\left[\frac{13\cos(v)\sin(v)}{10}\right]_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)} + \left[\frac{13}{20}v\right]_{\arcsin\left(\frac{2}{\sqrt{13}}\right)}^{\arcsin\left(\frac{3.6}{\sqrt{13}}\right)}$. Next, I undid substitution $v = \arcsin\left(\frac{2u}{\sqrt{13}}\right)$. $\left[\frac{13\cos\left(\arcsin\left(\frac{2u}{\sqrt{13}}\right)\right)\sin\left(\arcsin\left(\frac{2u}{\sqrt{13}}\right)\right)}{10}\right]_1^{1.8} + \left[\frac{13\arcsin\left(\frac{2u}{\sqrt{13}}\right)}{20}\right]_1^{1.8}$. Here, $\sin\left(\arcsin\left(\frac{2u}{\sqrt{13}}\right)\right) = \frac{2u}{\sqrt{13}}$ but $\cos\left(\arcsin\left(\frac{2u}{\sqrt{13}}\right)\right)$ is a bit more complex. To solve $\cos(\arcsin(x))$, I will use the triangle



where $\theta = \arcsin(x)$, $B = x$, $C = 1$, $A^2 = C^2 - B^2 = 1-x^2$, $A = \sqrt{1-x^2}$ so $\cos(\arcsin(x)) = \cos(\theta) = \frac{A}{C} = \sqrt{1-x^2}$ then $\cos\left(\arcsin\left(\frac{2u}{\sqrt{13}}\right)\right) = \sqrt{1-\frac{4u^2}{13}}$. I undid my substitutions and substituted my solved integrals into my original integral

$-\frac{1}{10}\int_{1.8}^{1.96}\sqrt{13-4(5x-8)^2}dx+\frac{13}{20}\int_{1.8}^{1.96}1\,dx$. The result is

$\left[-\frac{\sqrt{13}(5x-8)\sqrt{1-\frac{4(5x-8)^2}{13}}}{100}-\frac{13\arcsin\left(\frac{10x-16}{\sqrt{13}}\right)}{200}+\frac{13x}{20}\right]_{1.8}^{1.96}$. Originally, I had set an upper boundary 2 when graphing the function because this is a half-circle with a set domain. However, calculating the integral with upper bound 2 results in the complex number 0.096−0.005i, rounded to 3 significant figures. This is because the value $\frac{10\times2-16}{\sqrt{13}}=1.11$ is greater than 1, outside the domain of the $arcsin(x)$ function. The actual upper boundary value can be calculated by solving $\frac{10x-16}{\sqrt{13}}=1$. The approximation $x=1.96$ is slightly smaller than the actual value and substituting this as the upper boundary into the integral, the area is 0.0701.

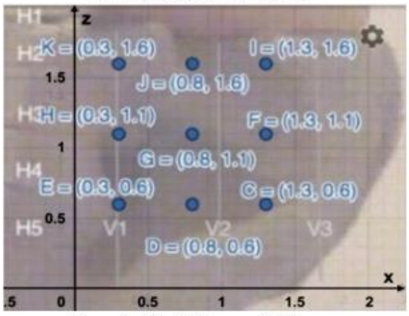
Next, I will show my work for integrating $h_1(x)=2sin(x-3)+4.9$ also in H4. First, I applied linearity $\int_{7.78}^{9}(2\sin(x-3)+4.9)dx=2\int_{7.78}^{9}\sin(x-3)dx+4.9\int_{7.78}^{9}1dx$ and used substitution $u=x-3$; $dx=du$; $u_l=4.78$; $u_u=6$. Then, I integrated $\int_{4.78}^{6}\sin(u)\,\mathrm{du}=[-\cos(\mathrm{u})]_{4.78}^{6}$. Substituting the solved integral into the original integral, the solution is $2[-\cos(\mathrm{u})]_{4.78}^{6}+4.9[x]_{7.78}^{9}=4.19$. Here, I used radians to solve the integral.

Next, to determine the similarity between my ear and my model, I filled the clay model with water to measure the actual volume, which was 5 ml and considering the accuracy of the measuring cylinder, I will assume the volume is in range $4cm^3$ to $6cm^3$. I calculated the volume of my model using the areas found by integration and adding or subtracting values depending on whether they were shifted up or down. For example, $\mathrm{H5}_{area}=1.827cm^2$ so the volume of my ear in range $0.3\le z\le0.4$ is $0.1827cm^3$ and volume in range $0.2\le z\le0.3$ is $(1.827cm^2-0.0215\times0.7+0.0599\times0.7+0.1018\times1.1)\times0.1cm=0.1966cm^3$. The values were taken from the shifting calculations on page 13. Other volumes are found similarly and added. $0.4545cm^3+0.6136cm^3+0.9208cm^3+1.200cm^3+0.8204cm^3=4.009cm^3$. I used four significant figures to get accurate values and because my 3D model isn't very accurate, it is much smaller than 5. However, it is within the range 4 to 6 and I will be able to use it as an approximate ear model.

3D Sound Wave Reflections

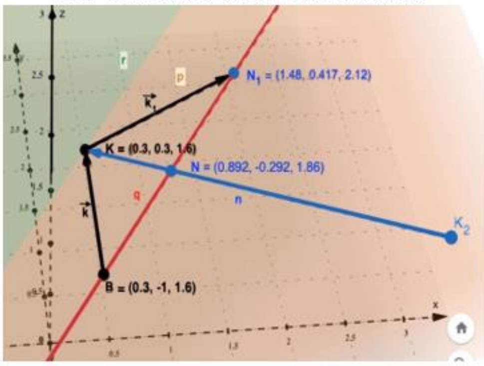
My next step is to calculate the reflections of the nine chosen sound waves as shown in Graph S. To make my work concise, I will only show my working for sound wave $\vec{k}$ which intersects with my ear at point K, which lies on function H2 and V1. The function I graphed for H2 is 9 times larger than my ear size. Therefore, point $x = 0.3$ is point $x = 2.7$ on function H2, and the ordinate of H2 is $f(2.7) = 2.7$. Dividing this value by 9 gives us $y = 0.3$.The equation of the sound wave is $\vec{k} = \begin{pmatrix} 0.3 \\ 0.3 \\ 1.6 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mu_1$. To determine where the sound wave will reflect, I drew a plane passing through point $K = (0.3, 0.3, 1.6)$ and two direction vectors. The first direction vector has the slope of the horizontal function 2 at point $x = 0.3$ where the z is constant. The function is $f(x) = x$ and its slope is$f'(x) = 1$. The direction vector is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \delta_1$. The other direction vector has a slope equal to the slope of the vertical function at point $z = 1.6$. The x value is $0 \leq 0.3 \leq 0.7$ so I will use function V1. The y value when $z = 1.6$ is $s(y) = -\sqrt{0.06 - (y - 0.17)^2} + 1.7 = 1.6$, $y = 0.394$. Applying the chain rule, the derivative is $s'(y) = \frac{y - 0.17}{\sqrt{0.06 - (y - 0.17)^2}}$ and the slope is $s'(0.394) = \frac{0.394 - 0.17}{\sqrt{0.06 - (0.394 - 0.17)^2}} = 2.26$. x is constant so the direction vector is $\begin{pmatrix} 0 \\ 1 \\ 2.26 \end{pmatrix} \gamma_1$ and the plane equation is $r = \begin{pmatrix} 0.3 \\ 0.3 \\ 1.6 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \delta_1 + \begin{pmatrix} 0 \\ 1 \\ 2.26 \end{pmatrix} \gamma_1$, the green plane in Graph T. Now, I need to calculate the

3D Sound Waves



Graph S. 9 Sound Waves

3D Sound Wave Reflection



Graph T. Sound Wave k and Reflection k_1

angle between this plane and the sound wave. First, I will find the normal to the plane passing through point K. The direction vector is the cross product of the two direction vectors of the plane. $\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2.26\end{pmatrix} = \begin{pmatrix}2.26\\-2.26\\1\end{pmatrix}$. The normal vector also passes through point K so $\vec{n} = \begin{pmatrix}0.3\\0.3\\1.6\end{pmatrix} + \begin{pmatrix}2.26\\-2.26\\1\end{pmatrix}\lambda_2$. To measure the angle between the normal vector and the sound wave vector, I will change the direction vector $\vec{n}$ so its direction is into the ear and I will use equation $\theta = \arccos\left(\frac{\vec{n}\cdot\vec{k}}{|\vec{n}|\times|\vec{k}|}\right)$; $\vec{n} = \begin{pmatrix}-2.26\\2.26\\-1\end{pmatrix}$; $\vec{k} = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $|\vec{n}| = 3.35$; $|\vec{k}| = 1$; $\vec{n}\cdot\vec{k} = 2.26$. Using angle $\theta = \arccos\left(\frac{2.26}{3.35}\right) = 47.6^o$, the angle between the plane and sound wave is $90^{\text{o}} - 47.6^o = 42.4^{\text{o}}$. Next, I need to find the vector equation of the reflecting sound wave. To do this, I will calculate the perpendicular from point $B = (0.3, -1, 1.6)$ to vector $\vec{n} = \begin{pmatrix}0.3\\0.3\\1.6\end{pmatrix} + \begin{pmatrix}2.26\\-2.26\\1\end{pmatrix}\lambda_2$ and its length. Let point $N = (0.3 + 2.26\lambda_2, 0.3 - 2.26\lambda_2, 1.6 + \lambda_2)$ be the point of intersection. Then, $\overrightarrow{BN} = \begin{pmatrix}2.26\lambda_2\\1.3 - 2.26\lambda_2\\\lambda_2\end{pmatrix}$. Vector $\vec{n}$ and $\overrightarrow{BN}$ are perpendicular so $2.26\lambda_2 \times 2.26 + (1.3 - 2.26\lambda_2) \times (-2.26) + \lambda_2 = 0$. Therefore, $\lambda_2 = 0.262$, $\overrightarrow{BN} = \begin{pmatrix}0.592\\0.708\\0.262\end{pmatrix}$ and the point of intersection is $N = (0.892, -0.292, 1.86)$ and the length of vector BN is 0.959. Now I need to find a point N_1 that is perpendicular to vector $\vec{n}$ and at a distance of 0.959 from it on the opposite side. $N_1 = (0.892 + 0.592, -0.292 + 0.708, 1.86 + 0.262)$ $N_1 = (1.48, 0.417, 2.12)$. The equation of the reflected sound wave is $\overrightarrow{k_1} = \begin{pmatrix}0.3\\0.3\\1.6\end{pmatrix} + \begin{pmatrix}1.18\\0.117\\0.522\end{pmatrix}\delta_2$.

$x = 0.3 + 1.18\delta_2$
$y = 0.3 + 0.117\delta_2$. Now, I will check if the sound wave enters the ear canal by checking if the
$z = 1.6 + 0.522\delta_2$

vector intersects one of the two planes within the given range. Equation for plane c_3 is $-0.9x - y = -2.85$ and equation for plane c_4 is $-0.66x - y = -2.54$. The expressions for x, y, and z are substituted into the two plane equations to calculate the point of intersection. Calculation for first intersection $-0.9(0.3 + 1.18\delta_2) - (0.3 + 0.117\delta_2) = -2.85$ and $\delta_2 = 1.93$ so $C_3 = (2.58, 0.526, 2.61)$, which is not in range $0.578 \leq x \leq 0.8$, $0.944 \leq y \leq 1.14$, $0.8 \leq z \leq 1.2$. The point of intersection with the second plane is $C_4 = (2.99, 0.569, 2.80)$ which is not in range $0.255 \leq x \leq 1.09$, $1.08 \leq y \leq 1.62$, $0.4 \leq z \leq 0.8$. Therefore, the sound wave does not enter the ear canal. From my calculations, the 5 sound waves h, f, e, d, and c out of 9 enter the ear canal. Once again, these five sound waves reflected off the same region of the ear.

Conclusion

The purpose of this exploration was to investigate the effectiveness of the shape of the auricle at reflecting sound waves into the ear canal. I had a total of three methods, with success rates 50%, 17%, and 56% respectively. While my first two methods had their limitations, they enabled me to investigate my question with a simple model. I used a constant number of parallel sound waves evenly spread to ensure the reliability of my data. The limitation for my first two models was resolved by my third and final method.

The limitation of my third model was caused by the simplifications made to approximate the ear shape. I was able to calculate changes in the vertical slope to shift the horizontal functions and combine them into a 3D piecewise function. However, there is an inaccuracy because I used only three vertical slopes and the regions where different functions meet does not match. Also, due to the complexity of the calculations, I only calculated the first reflection, not possible second or third reflections. Another source of error is in using clay ear models, as they are imperfect representations of my ear and there may be errors in stamping the cross sections leading to inaccuracies in my calculations.

I realized applying mathematics to real life scenarios is not simply solving problems but finding ways to represent scenarios mathematically using models. Throughout this exploration, this was a challenge to me as I sought different models I could use and to apply the mathematics I knew to investigate the effectiveness of the auricle. I also conducted research into solving matrices and spent a considerable amount of time calculating the reflections of sound waves and integrals of the functions to calculate the area. Furthermore, when checking my work, I realized the importance of rewriting and meticulously checking all my calculations one by one to ensure they are precise because when reading through the exploration, it is possible to miss small errors. Furthermore, it is important to make sure the results make sense and to examine any irregular answers to make sure they are not caused by errors. Working on this investigation greatly deepened my appreciation for mathematics and its applications to real life.

Overall, my methods were successful, and I was able to show how the unique shape of the auricle can funnel sound waves into the ear canal. To improve my method, I can graph more horizontal and vertical cross sections of the ear and calculate the reflections of more sound waves coming from different angles. Another interesting extension would be to model the ears of babies as they grow to learn how the ear develops. This may help us create more accurate synthetic ears. But until evolution evolves our auricle into a more effective shape, we can continue appreciating the wondrous sounds of our world with our naturally but effectively shaped auricles.

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